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Generalized Variable-Coefficient KP Equation

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The variable-coefficient generalizations of the celebrated KP equation (GvcKPs) are realistic models for various physical and engineering situations. In this note, the application of symbolic computation and the truncated Painlevé expansion leads to an auto-Bäcklund transformation and soliton-typed solutions to a type of the GvcKPs.

The variable-coefficient generalizations of the celebrated KP equation are of practical value since they are able to realistically model various engineering and physical situations. However, they are not so easy to be learnt, also because of the existence of their coefficient functions.

In this paper, the following type of the generalized variable-coefficient KP equation,

$$(u_t + uu_x + u_{xxx})_x + a(y, t)u_x + b(y, t)u_y + c(y, t)u_{yy} + d(y, t)u_{xy} + e(y, t)u_{xx} = 0, \tag{1}$$

is under investigation, where $a(y, t)$, $b(y, t)$, $c(y, t) \neq 0$, $d(y, t)$ and $e(y, t)$ are analytic, sufficiently differentiable functions. It has been shown (Clarkson, 1990) that the constraints which the variable-coefficient functions must satisfy for Eqn. 1 to pass the Painlevé tests for complete integrability are precisely the same as those in order that Eqn. 1 may be transformed into either the KP or KdV equations, thus becoming the necessary and sufficient conditions for Eqn. 1 to be completely integrable.

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We hereby do not make the assumption that Eqn. 1 possesses the Painleve property, but truncate the Painleve expansion (Weiss et al., 1983) at the constant level term, i.e.,

$$u(x, y, t) = \phi^{-J}(x, y, t) \sum_{l=0}^{J} u_l(x, y, t)\phi^l(x, y, t) \quad \text{with} \quad J = 2, \tag{2}$$

so as to obtain the Bäcklund transformation as well as some special solutions for Eqn. 1 without requiring it to be completely integrable. Hereby $J = 2$ is determined via the leading-order analysis, $u_l(x, y, t)$ and $\phi(x, y, t)$ are analytic functions with $u_0(x, y, t) \neq 0$.

When substituting Expression 2 into Eqn. 1 with symbolic computation, we make the coefficients of like powers of ϕ to vanish, so as to get the set of Painleve-Bäcklund (PB) equations. After symbolic computation, we find the following auto-Bäcklund transformation,

Eqn. 2,

$$u_0 = -12\phi_x^2, \qquad v_1 = 12\phi_{xx}, \qquad \phi_x \neq 0, \tag{3}$$

$$c\phi_y^2 + d\phi_x\phi_y + e\phi_x^2 + \phi_x\phi_t + \phi_x^2 u_2 - 3\phi_{xx}^2 + 4\phi_x\phi_{xxx} = 0, \tag{4}$$

$$(u_{2,t} + u_2 u_{2,x} + u_{2,xxx})_x + a u_{2,x} + b u_{2,y} + c u_{2,yy} + d u_{2,xy} + e u_{2,xx} = 0, \tag{5}$$

$$a\phi_x + b\phi_y + c\phi_{yy} + d\phi_{xy} + e\phi_{xx} + \phi_{xt} + \phi_{xxxx} + \phi_{xx} u_2 = 0. \tag{6}$$

Having made some assumptions and used symbolic computation to deal with complicated calculations, we end up with the following soliton-typed solutions:

$$u(x, y, t) = 3\beta^2 \operatorname{sech}^2\left\{\frac{\beta}{2} x + \frac{1}{2} \int e^{-\int [b(y,t)/c(y,t)]\, dy} \times \left[\gamma(t) - \beta \int \frac{a(y, t)}{c(y, t)} e^{\int [b(y,t)/c(y,t)]\, dy}\, dy\right] dy + \frac{\sigma(t)}{2}\right\} + \alpha, \tag{7}$$

where α and $\beta \neq 0$ are real constants, while $\gamma(t)$ and $\sigma(t)$ are real, differentiable functions. Those solutions hold when the following constraint on the coefficient functions is satisfied,

$$\beta^2 e^{\int [b(y,t)/c(y,t)]\, dy} d(y, t) \int \frac{a(y, t)}{c(y, t)} e^{\int [b(y,t)/c(y,t)]\, dy}\, dy$$
$$+ 2\beta c(y, t)\gamma(t) \int \frac{a(y, t)}{c(y, t)} e^{\int [b(y,t)/c(y,t)]\, dy}\, dy - \alpha\beta^2 e^{2\int [b(y,t)/c(y,t)]\, dy}$$
$$- \beta^4 e^{2\int [b(y,t)/c(y,t)]\, dy} - \beta^2 e^{2\int [b(y,t)/c(y,t)]\, dy} e(y, t)$$

$$
\begin{aligned}
&- \beta e^{\int [b(y,t)/c(y,t)]\, dy} d(y, t)\gamma(t) - c(y, t)\gamma^2(t) \\
&- \beta^2 c(y, t)\left[\int \frac{a(y, t)}{c(y, t)}\, e^{\int [b(y,t)/c(y,t)]\, dy}\, dy\right]^2 - \beta e^{2\int [b(y,t)/c(y,t)]\, dy}\sigma_t(t) \\
&+ \beta e^{2\int [b(y,t)/c(y,t)]\, dy} \int e^{-\int [b(y,t)/c(y,t)]\, dy} \\
&\cdot \left\{\beta \int c^{-2}(y, t) e^{\int [b(y,t)/c(y,t)]\, dy} \left[a(y, t)c(y, t) \right.\right. \\
&\cdot \int \frac{c(y, t)b_t(y, t) - b(y, t)c_t(y, t)}{c^2(y, t)}\, dy + c(y, t)a_t(y, t) \\
&\left. - a(y, t)c_t(y, t) \right] dy + \left[\gamma(t) - \beta \int \frac{a(y, t)}{c(y, t)}\, e^{\int [b(y,t)/c(y,t)]\, dy}\, dy\right] \\
&\left.\cdot \int \frac{c(y, t)b_t(y, t) - b(y, t)c_t(y, t)}{c^2(y, t)}\, dy - \gamma_t(t)\right\} dy = 0. \qquad (8)
\end{aligned}
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